

# STRANGE ATTRACTORS IN PERIODICALLY-KICKED DEGENERATE HOPF BIFURCATIONS

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**ABSTRACT.** We prove that spiral sinks (stable foci of vector fields) can be transformed into strange attractors exhibiting sustained, observable chaos if subjected to periodic pulsatile forcing. We show that this phenomenon occurs in the context of periodically-kicked degenerate supercritical Hopf bifurcations. The results and their proofs make use of a new  $k$ -parameter version of the theory of rank one maps developed by Wang and Young.

## 1. INTRODUCTION

This paper aims to support the idea that shear and twist are natural mechanisms for the production of sustained, observable chaos in forced dynamical systems. Consider a weakly stable dynamical structure such as an equilibrium point or a limit cycle. If shear or twist is present, then forcing of various types can transform the weakly stable structure into a strange attractor. The nature of the forcing is not essential. Admissible types of forcing include periodic pulsatile drives, deterministic continuous-time signals, and random signals generated by stochastic processes. The strange attractors possess many of the dynamical, statistical, and geometrical properties commonly associated with chaotic dynamics.

We study the simplest weakly stable dynamical structure. This is the spiral sink (or stable focus), an equilibrium point of a vector field with the property that the linearization of the field at the equilibrium point has a pair of complex conjugate eigenvalues  $\alpha \pm i\beta$  satisfying  $\alpha < 0$  and  $\beta \neq 0$ . We consider the degenerate supercritical Hopf bifurcation in two dimensions. When a generic supercritical Hopf bifurcation occurs, the spiral sink becomes unstable and a limit cycle is born. In the degenerate case, the spiral sink loses its stability but no limit cycle is born. We prove that in the case of the degenerate supercritical Hopf bifurcation, periodic pulsatile drives transform the spiral sink into a strange attractor. The analysis is certainly not limited to Hopf bifurcations. We work in this context because the origin of the shear is transparent in the defining differential equations.

The analysis is based on the beautiful dynamical theory of rank one maps formulated by Wang and Young [8, 7]. Speaking impressionistically, rank one maps are strongly dissipative maps exhibiting a single direction of instability. Rank one theory provides checkable conditions that imply the existence of strange attractors for a positive-measure set of parameters within a given parametrized family of rank one maps. The conditions appear within the following scheme.

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- (1) Let dissipation go to infinity. This procedure produces the singular limit, a parametrized family of one-dimensional maps.
- (2) Check that the singular limit includes a map with strong expanding properties (a map of Misiurewicz type).
- (3) Verify a parameter transversality condition.
- (4) Verify a nondegeneracy condition. This allows information about the singular limit to be passed to the maps with finite dissipation.

Steps (3) and (4) cumulatively require verifying that only finitely many quantities do not vanish. For good parameters, parameters corresponding to maps admitting strange attractors, rank one theory provides a reasonably complete dynamical description of the map.

The attractor supports a positive, finite number of ergodic SRB measures. The orbit of Lebesgue almost-every point in the basin of attraction has a positive Lyapunov exponent and is asymptotically distributed according to one of the ergodic SRB measures. Each SRB measure satisfies the central limit theorem and exhibits exponential decay of correlations. A symbolic coding exists for orbits on the attractor. This symbolic coding implies the existence of equilibrium states and a measure of maximal entropy. Summarizing, the map has a nonuniformly hyperbolic character and exhibits sustained, observable chaos.

Of the four steps in the rank one scheme, step (2) is the most fundamental and typically requires the most work. A Misiurewicz map has the property that the positive orbit of *every* critical point remains bounded away from the critical set. Existing papers on rank one theory view the singular limit as a one-parameter family  $\{f_a\}$  of one-dimensional maps. This view makes locating Misiurewicz parameters difficult if the maps have multiple critical points. If each map  $f_a$  has exactly one critical point  $c(a)$ , then locating Misiurewicz parameters is relatively easy. Assuming that  $f_a(c(a))$  moves reasonably quickly as one varies  $a$ , simply locate an invariant set  $\Lambda(a)$  that is disjoint from the critical set and then choose  $a^*$  such that  $f_{a^*}(c(a^*)) \in \Lambda(a^*)$ . The set  $\Lambda(a)$  could be a periodic orbit or a Cantor set. If the singular limit consists of maps with multiple critical points, then one must locate a parameter  $a^*$  for which all of the critical orbits of  $f_{a^*}$  are contained in good invariant sets. This is a serious challenge because good invariant sets such as periodic orbits and Cantor sets typically have Lebesgue measure zero. Wang and Young [9] overcome this challenge. However, their results assume that the maps in the singular limit possess an extremely large amount of expansion.

We prove that significantly less expansion is needed if the singular limit is viewed as an  $m$ -parameter family for  $m$  sufficiently large. Assume that the singular limit consists of maps with  $k$  critical points. We prove that if the singular limit is viewed as a  $k$ -parameter family, then it contains Misiurewicz points assuming the maps are mildly expanding and assuming the parameters are independent in a sense to be made precise. This result widens the scope of rank one theory.

We view this work as an element of a growing list of applications of rank one theory. The theory has been rigorously applied to simple mechanical systems [9], periodically-kicked limit cycles and Hopf bifurcations [10], and the Chua circuit [6]. Guckenheimer, Wechselberger, and Young [1] connect rank one theory and geometric singular perturbation theory by formulating a general technique for proving the existence of chaotic attractors for three-dimensional vector fields with two time

scales. Lin [2] demonstrates how rank one theory can be combined with sophisticated computational techniques to analyze the response of concrete nonlinear oscillators of interest in biological applications to periodic pulsatile drives. Lin and Young [3] study shear-induced chaos numerically in situations beyond the reach of current analytical tools. In particular, they consider stochastic forcing. This work supports the belief that shear-induced chaos is both widespread and robust.

We organize the presentation of ideas as follows. In Section 2, we present the main results for periodically-kicked degenerate Hopf bifurcations. In Section 3, we prove the result concerning the existence of Misiurewicz points in  $k$ -parameter families of one-dimensional maps and we present a two-parameter example. Section 4 presents rank one theory viewing the singular limit as a  $k$ -parameter family. Finally, in Section 5 we prove the results presented in Section 2.

## 2. PERIODICALLY-KICKED DEGENERATE HOPF BIFURCATIONS

The normal form for the supercritical Hopf bifurcation in two spatial dimensions is given in polar coordinates by

$$\begin{cases} \dot{r} = (\mu - \alpha_\mu r^2)r + r^5 g_\mu(r, \theta) \\ \dot{\theta} = \omega + \gamma_\mu \mu + \beta_\mu r^2 + r^4 h_\mu(r, \theta) \end{cases}$$

Here  $\mu$  is the bifurcation parameter and  $\omega$  is a constant. The multipliers  $\alpha_\mu$ ,  $\gamma_\mu$ , and  $\beta_\mu$  depend smoothly on  $\mu$ . The functions  $g_\mu$  and  $h_\mu$  depend smoothly on  $\mu$  and they are of class  $C^4$  with respect to  $r$  and  $\theta$ . The normal form for the degenerate Hopf bifurcation in two spatial dimensions is obtained by setting  $\alpha_\mu = 0$  for all  $\mu$  and replacing  $\mu$  with  $-\mu$ , yielding

$$(2.1) \quad \begin{cases} \dot{r} = -\mu r + r^5 g_\mu(r, \theta) \\ \dot{\theta} = \omega + \gamma_\mu \mu + \beta_\mu r^2 + r^4 h_\mu(r, \theta) \end{cases}$$

For  $\mu > 0$ , the origin is an asymptotically stable equilibrium point (a sink). We study (2.1) in this  $\mu$ -range. Let  $\widehat{F}_t$  denote the flow generated by (2.1). We perturb the flow  $\widehat{F}_t$  with a ‘kick’ map  $\kappa$  defined as follows. Let  $L > 0$  and let  $\rho_2 > 0$ . The map  $\kappa = \kappa_{\mu, L, \rho_2}$  is given in rectangular coordinates by

$$\kappa \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) + L\mu^{\rho_2} \end{pmatrix}$$

The composition  $\widehat{F}_t \circ \kappa$  may be thought of as a perturbation followed by a period of relaxation. We define an annulus map associated with  $\widehat{F}_t \circ \kappa$ . Let  $\mathcal{A}$  denote the annulus defined by

$$\mathcal{A} = \{(r, \theta) : K_4^{-1}\mu^{\rho_1} \leq r \leq K_4\mu^{\rho_1}\},$$

where  $K_4 > 1$  and  $0 < \rho_2 < \rho_1$ . Let  $\tilde{r}$  denote the distance from  $\kappa(\mathcal{A})$  to the origin. We have  $\tilde{r} = L\mu^{\rho_2} - K_4\mu^{\rho_1}$ . Define the relaxation time  $\tau(\mu)$  by

$$\tilde{r}e^{-\mu\tau(\mu)} = \mu^{\rho_1}.$$

For  $\mu$  sufficiently large,  $\widehat{F}_{\tau(\mu)} \circ \kappa$  maps  $\mathcal{A}$  into  $\mathcal{A}$ .

The following theorem states that under certain conditions, the annulus map  $\widehat{F}_{\tau(\mu)} \circ \kappa$  admits a strange attractor for a positive-measure set of values of  $\mu$ . We make the crucial assumption that the twist factor  $\beta_0$  is nonzero. A nonzero twist factor implies the existence of an angular-velocity gradient in the radial direction

for values of  $\mu$  in a neighborhood of the bifurcation parameter  $\mu = 0$ . This angular-velocity gradient allows the flow to stretch and fold the phase space, thereby producing chaos.

The chaos in this setting is sustained in time and observable. The strange attractors possess many of the geometric and dynamical properties normally associated with chaotic systems. These properties include the existence of a positive Lyapunov exponent **(SA1)**, the existence of SRB measures and basin property **(SA2)**, and statistical properties such as exponential decay of correlations and the central limit theorem for dynamical observations **(SA3)**. In addition, if  $\frac{L}{|\beta_0|}$  is sufficiently large, then the annulus map admits a unique SRB measure **(SA4)**. Properties **(SA1)**-**(SA4)** are described in detail in Section 4.

**Theorem 2.1.** *Assume  $\beta_0 \neq 0$ . Let  $\rho_1$  and  $\rho_2$  satisfy  $\rho_2 \in (\frac{3}{8}, \frac{1}{2})$  and  $\rho_1 + \rho_2 = 1$ .*

- (1) *There exists  $M_0 > 0$  such that if  $L \geq \frac{M_0}{|\beta_0|}$ , then there exist  $L^* \in [L, L + \frac{\pi}{|\beta_0|}]$  and  $\mu_0 > 0$  satisfying the following. The parameter interval  $(0, \mu_0]$  contains a set  $\Delta = \Delta(L^*)$  of positive measure such that for  $\mu \in \Delta$ , the map  $\hat{F}_{\tau(\mu)} \circ \kappa$  admits a strange attractor with properties **(SA1)**, **(SA2)**, and **(SA3)**. The set  $\Delta$  intersects every interval of the form  $(0, \tilde{\mu}]$  in a set of positive measure.*
- (2) *There exists  $M_1 \gg M_0$  such that for all  $L \geq \frac{M_1}{|\beta_0|}$ , there exists a set  $\Delta = \Delta(L)$  with the properties described in (1).*
- (3) *If  $L$  is sufficiently large and  $\mu \in \Delta(L)$ , then **(SA4)** holds as well.*

### 3. LOCATING MISIUREWICZ POINTS

Let  $I$  denote an interval or the circle  $S^1$ . Let  $F : I \times [a_1, a_2] \times [b_1, b_2] \rightarrow I$  be a  $C^2$  map. The map  $F$  defines a two-parameter family  $\mathcal{F} = \{f_{a,b} : a \in [a_1, a_2], b \in [b_1, b_2]\}$  via  $f_{a,b}(x) = F(x, a, b)$ . Set  $A = [a_1, a_2]$  and  $B = [b_1, b_2]$ . We assume that for each  $(a, b) \in A \times B$ ,  $f_{a,b}$  has two critical points. We label these critical points  $c^{(1)}(a, b)$  and  $c^{(2)}(a, b)$ . Let  $C = C(a, b) = \{c^{(1)}(a, b), c^{(2)}(a, b)\}$ . For  $\delta > 0$ , let  $C_\delta$  denote the  $\delta$ -neighborhood of  $C$  in  $I$ .

We seek to identify conditions under which  $\mathcal{F}$  contains strongly expanding (Misiurewicz) maps. We now introduce this class.

**Definition 3.1.** We say that  $f \in C^2(I, I)$  is a Misiurewicz map and we write  $f \in \mathcal{M}$  if the following hold for some neighborhood  $V$  of  $C$ .

- (A) **(Outside of  $V$ )** There exist  $\lambda_0 > 0$ ,  $M_0 \in \mathbb{Z}^+$ , and  $0 < d_0 \leq 1$  such that
  - (1) for all  $n \geq M_0$ , if  $f^k(x) \notin V$  for  $0 \leq k \leq n-1$ , then  $|(f^n)'(x)| \geq e^{\lambda_0 n}$ ,
  - (2) for any  $n \in \mathbb{Z}^+$ , if  $f^k(x) \notin V$  for  $0 \leq k \leq n-1$  and  $f^n(x) \in V$ , then  $|(f^n)'(x)| \geq d_0 e^{\lambda_0 n}$ .
- (B) **(Critical orbits)** For all  $c \in C$  and  $n > 0$ ,  $f^n(c) \notin V$ .
- (C) **(Inside  $V$ )**
  - (1) We have  $f''(x) \neq 0$  for all  $x \in V$ , and
  - (2) for all  $x \in V \setminus C$ , there exists  $p_0(x) > 0$  such that  $f^j(x) \notin V$  for all  $j < p_0(x)$  and  $|(f^{p_0(x)})'(x)| \geq d_0^{-1} e^{\frac{1}{3} \lambda_0 p_0(x)}$ .

We first formulate hypotheses that imply the existence of maps in  $\mathcal{F}$  that satisfy Definition 3.1(B).

**3.1. The general result.** We formulate the result for two-parameter families consisting of maps with two critical points. The result generalizes in a natural way for  $k$ -parameter families consisting of maps with  $k$  critical points.

The first hypothesis is formulated in terms of the evolutions

$$(a, b) \mapsto \gamma_n^{(i)}(a, b) \text{ where } \gamma_n^{(i)}(a, b) = f_{a,b}^n(c^{(i)}(a, b)).$$

The evolutions  $\{\gamma_n^{(i)} : n \in \mathbb{N}\}$  generate **critical curve** dynamics. Define  $\Gamma_n : A \times B \rightarrow I \times I$  by  $\Gamma_n = (\gamma_n^{(1)}, \gamma_n^{(2)})$ .

We now present the general hypotheses. For  $J \subset I$  and  $\varepsilon > 0$ , let  $J^\varepsilon$  denote the  $\varepsilon$ -neighborhood of  $J$ . Suppose there exist subintervals  $I_1$  and  $I_2$  of  $I$ , subintervals  $\Delta_1 \subset A$  and  $\Delta_2 \subset B$ ,  $\delta_1 > 0$ , and  $\varepsilon_1 > 0$  such that the following hold.

- (H1) **(Finite Misiurewicz condition)** There exists  $n_0 \in \mathbb{Z}^+$  such that  $\Gamma_{n_0}(\Delta_1 \times \Delta_2) \supset I_1 \times I_2$  and for  $i \in \{1, 2\}$ ,  $(a, b) \in \Delta_1 \times \Delta_2$ , and  $n < n_0$ , we have  $\gamma_n^{(i)}(a, b) \in I \setminus C_{\delta_1}(a, b)$ .
- (H2) There exist fixed parameters  $\hat{a} \in \Delta_1$  and  $\hat{b} \in \Delta_2$  satisfying  $f_{\hat{a}, \hat{b}}(I_1) \times f_{\hat{a}, \hat{b}}(I_2) \supset I_1^{\varepsilon_1} \times I_2^{\varepsilon_1}$ .
- (H3) For all  $(a, b) \in \Delta_1 \times \Delta_2$ , we have  $I_1 \times I_2 \subset I \setminus C_{\delta_1}(a, b) \times I \setminus C_{\delta_1}(a, b)$ .

**Proposition 3.2.** *Suppose  $\mathcal{F}$  satisfies (H1)-(H3). If*

$$(3.1) \quad 2\sqrt{2} \max\{\|\partial_a F\|_{C^0}, \|\partial_b F\|_{C^0}\} \cdot \max\{|\Delta_1|, |\Delta_2|\} < \varepsilon_1,$$

*then there exists  $(a^*, b^*) \in \Delta_1 \times \Delta_2$  such that for  $i \in \{1, 2\}$  and for every  $n \in \mathbb{N}$ ,  $\gamma_n^{(i)}(a^*, b^*) \in I \setminus C_{\delta_1}(a^*, b^*)$ .*

*Proof of Proposition 3.2.* Define  $G = (f_{\hat{a}, \hat{b}}, f_{\hat{a}, \hat{b}})$ . Applying (H1) and (H2), we have  $G(\Gamma_{n_0}(\Delta_1 \times \Delta_2)) \supset I_1^{\varepsilon_1} \times I_2^{\varepsilon_1}$ . For every  $(a, b) \in \Delta_1 \times \Delta_2$ , we have

$$\|\Gamma_{n_0+1}(a, b) - G(\Gamma_{n_0}(a, b))\| < \varepsilon_1$$

since  $\varepsilon_1$  satisfies (3.1). Therefore,  $\Gamma_{n_0+1}(\Delta_1 \times \Delta_2) \supset I_1 \times I_2$ . Inductively,  $\Gamma_n(\Delta_1 \times \Delta_2) \supset I_1 \times I_2$  for all  $n \geq n_0$ . Define

$$\Psi_{n_0} = (\Delta_1 \times \Delta_2) \cap \Gamma_{n_0}^{-1}(I_1 \times I_2).$$

For  $n > n_0$ , define

$$\Psi_{n+1} = \Psi_n \cap \Gamma_{n+1}^{-1}(I_1 \times I_2).$$

Let

$$\Psi = \bigcap_{k=n_0}^{\infty} \Psi_k$$

and choose  $(a^*, b^*) \in \Psi$ . ■

**3.2. Verifying (H1).** We present a two-step procedure for the verification of hypothesis H1. First, we assume that  $\Gamma_1$  is a diffeomorphism on  $\Delta_1 \times \Delta_2$ . This implies that the image of  $\Delta_1 \times \Delta_2$  contains a rectangle in  $I \times I$ . Second, if we assume that each map  $f_{a,b}$  is expanding on  $I \setminus C_{\delta_1}$ , then the evolutions  $\gamma^{(1)}$  and  $\gamma^{(2)}$  will enlarge the rectangle to macroscopic size. The required time for this enlargement depends upon the magnitude of the expansion. Therefore, greater expansion results in a smaller value of  $n_0$ . We now make these ideas precise.

Suppose that  $\Gamma_1$  is a diffeomorphism on  $\Delta_1 \times \Delta_2$  such that for  $i \in \{1, 2\}$  and for every  $(a, b) \in \Delta_1 \times \Delta_2$ , we have  $\gamma_1^{(i)}(a, b) \in I \setminus C_{\delta_1}$ . Define

$$J(a, b) = \begin{vmatrix} \partial_a \gamma_1^{(1)}(a, b) & \partial_b \gamma_1^{(1)}(a, b) \\ \partial_a \gamma_1^{(2)}(a, b) & \partial_b \gamma_1^{(2)}(a, b) \end{vmatrix}.$$

Assume that there exists  $k_0 > 0$  such that  $|J| \geq k_0$  on  $\Delta_1 \times \Delta_2$ . This implies that  $\Gamma_1(\Delta_1 \times \Delta_2)$  contains a box with side length bounded below by

$$\frac{k_0^2}{\lambda_M} \min\{|\Delta_1|, |\Delta_2|\},$$

where

$$\lambda_M = \sup_{(a,b) \in \Delta_1 \times \Delta_2} \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } D\Gamma_1^* D\Gamma_1\}.$$

Now suppose that for every  $(a, b) \in \Delta_1 \times \Delta_2$  we have  $|f'_{a,b}| \geq K > 1$  on  $I \setminus C_{\delta_1}$ . Lower bounds on  $K$  will be given as the discussion proceeds.

We choose  $K$  based on the magnitudes of the partial derivatives of  $\gamma_1^{(1)}$  and  $\gamma_1^{(2)}$ . Assume there exists  $\rho > 0$  such that on  $\Delta_1 \times \Delta_2$  we have

- (1)  $|\partial_a \gamma_1^{(1)}| \geq \rho$  or  $|\partial_b \gamma_1^{(1)}| \geq \rho$ , and
- (2)  $|\partial_a \gamma_1^{(2)}| \geq \rho$  or  $|\partial_b \gamma_1^{(2)}| \geq \rho$ .

Suppose for the sake of definiteness that (1) and (2) hold with respect to the operator  $\partial_a$ . We now relate spatial and parametric derivatives. The equation

$$(3.2) \quad \frac{\partial}{\partial a} \gamma_{n+1}^{(i)}(a, b) = f'_{a,b}(\gamma_n^{(i)}(a, b)) \cdot \frac{\partial}{\partial a} \gamma_n^{(i)}(a, b) + \frac{\partial F}{\partial a}(\gamma_n^{(i)}(a, b), a, b)$$

implies that parametric derivatives grow exponentially provided that spatial derivatives grow exponentially. If  $K$  satisfies

$$K\rho - \|\partial_a F\|_{C^0} \geq \frac{3}{4}K, \text{ and}$$

$$\|\partial_a F\|_{C^0} \sum_{j=2}^{\infty} K^{-j} \leq \frac{1}{4},$$

then (3.2) implies that

$$(3.3) \quad |\partial_a \gamma_n^{(i)}(a, b)| \geq \frac{1}{2}K^n$$

provided  $\gamma_j^{(i)}(a, b) \in I \setminus C_{\delta_1}$  for  $j < n$ . Hypothesis H1 may be verified as follows. Look for a time  $n_0$  such that for  $i \in \{1, 2\}$ ,  $\gamma_{n_0}^{(i)}(\Delta_1 \times \{b\}) \supset I_i$  for every  $b \in \Delta_2$  and  $\gamma_k^{(i)}(a, b) \in I \setminus C_{\delta_1}$  for all  $k < n_0$  and  $(a, b) \in \Delta_1 \times \Delta_2$ . By (3.3), we have

$$K^{n_0} \approx \frac{2\lambda_M \max\{|I_1|, |I_2|\}}{k_0^2 \min\{|\Delta_1|, |\Delta_2|\}}.$$

**3.3. A two-parameter example.** Let  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . Let  $\Phi : S^1 \rightarrow \mathbb{R}$  be a  $C^3$  function with two nondegenerate critical points  $c^{(1)}$  and  $c^{(2)}$ . We assume that  $\Phi(c^{(1)}) \neq \Phi(c^{(2)})$ . Fix  $\zeta \in S^1$ . Consider the two-parameter family of circle maps  $\mathcal{F} = \{f_{a,L} : a \in S^1, L \in \mathbb{R}^+\}$  defined by

$$f_{a,L}(\theta) = \zeta + L\Phi(\theta) + a.$$

Small perturbations of this family frequently arise as singular limits of rank one families.

**Definition 3.3.** We say that  $(a, L)$  is a *Misiurewicz pair* if  $f_{a,L} \in \mathcal{M}$ .

The goal of this subsection is to prove the following result.

**Theorem 3.4.** *There exists  $L_0 > 0$  such that if  $L \geq L_0$ , then there exists a Misiurewicz pair  $(a^*, L^*)$  with  $L^* \in [L, L + 2\pi/|\Phi(c^{(2)}) - \Phi(c^{(1)})|]$  and  $a^* \in [0, 2\pi)$ .*

**Remark 3.5.** Misiurewicz points occur with greater frequency as  $L$  increases. Wang and Young [9] prove that there exists  $L_1 \gg L_0$  such that if  $L \geq L_1$ , then  $f_{a,L} \in \mathcal{M}$  for a  $\mathcal{O}(1/L)$ -dense subset of parameters  $a \in [0, 2\pi)$ .

*Proof of Theorem 3.4.* We prove Theorem 3.4 in two steps. We first show that for  $L$  sufficiently large, if  $f_{a,L}$  satisfies Definition 3.1(B), then  $f_{a,L} \in \mathcal{M}$ . We then prove the existence of parameters for which  $f_{a,L}$  satisfies Definition 3.1(B). Set  $f = f_{a,L}$  for the sake of simplicity.

Let  $k_1 = \frac{1}{2} \min\{\Phi''(c^{(1)}), \Phi''(c^{(2)})\}$ . There exists  $\delta_2 = \delta_2(\Phi)$  such that  $|c^{(2)} - c^{(1)}| > 2\delta_2$  and  $|\Phi''| > k_1$  on  $C_{\delta_2}$ . Notice that  $|f''| > k_1 L$  on  $C_{\delta_2}$ . At this point we introduce the auxiliary constant  $K$ . This constant will be used to bound the derivative of  $f$  from below away from the critical set. Lower bounds on  $K$  will be given as the proof develops. We choose  $K$  before we choose  $L$ . Let  $\sigma = 2k_1^{-1}L^{-1}K^3$  and assume  $\frac{\sigma}{2} < \delta_2$ . For  $x \in C_{\delta_2} \setminus C_{\frac{1}{2}\sigma}$  we have  $|f'(x)| \geq K^3$ . Choose  $L$  sufficiently large so that  $|f'| \geq K^3$  outside  $C_{\frac{1}{2}\sigma}$ . Summarizing, the map  $f$  has the following properties.

(P1)  $|f''| > k_1 L$  on  $C_{\delta_2}$

(P2)  $|f'| \geq K^3$  outside  $C_{\frac{1}{2}\sigma}$

The following recovery lemma asserts that if an orbit visits a small neighborhood of a critical point, then the derivative along this orbit regains a definite amount of exponential growth as this orbit tracks the orbit of the critical point for a period of time. Set  $K_2 = \|\Phi\|_{C^2}$ . Let  $V = \{x \in S^1 : |f'(x)| \leq K\}$  and note that  $V \subset C_{\frac{1}{2}\sigma}$ . Together with (P1) and (P2), Lemma 3.6 implies that if  $f$  satisfies Definition 3.1(B), then  $f \in \mathcal{M}$ .

**Lemma 3.6** (Recovery estimate). *Let  $c \in C$  be such that  $f^n(c) \notin C_\sigma$  for all  $n \in \mathbb{N}$ . For  $x \in V$ , let  $n(x)$  be the smallest value of  $n$  such that  $|f^n(x) - f^n(c)| > \frac{1}{4K_2}K^3L^{-1}$ . We have  $n(x) > 1$  and  $|(f^{n(x)})'(x)| \geq k_3K^{n(x)}$  for some  $k_3 = k_3(k_1, K_2)$ .*

The proof of Lemma 3.6 uses the following distortion estimate.

**Sublemma 3.7** (Local distortion estimate). *Let  $x, y \in S^1$ . For  $i \in \mathbb{Z}^+$ , let  $\omega_i$  denote the segment between  $f^i(x)$  and  $f^i(y)$ . If  $n \in \mathbb{Z}^+$  is such that  $|\omega_i| \leq \frac{1}{4K_2}K^3L^{-1}$  and  $d(\omega_i, C) \geq \frac{1}{2}\sigma$  for all  $0 \leq i < n$ , then  $\frac{(f^n)'(x)}{(f^n)'(y)} \leq 2$ .*

*Proof of Sublemma 3.7.* We have

$$\begin{aligned}
\log \left( \frac{(f^n)'(x)}{(f^n)'(y)} \right) &= \sum_{i=0}^{n-1} \log \left( \frac{f'(f^i(x))}{f'(f^i(y))} \right) \\
&\leq \sum_{i=0}^{n-1} \frac{|f'(f^i(x)) - f'(f^i(y))|}{|f'(f^i(y))|} \\
&\leq \sum_{i=0}^{n-1} \frac{LK_2 |f^i(x) - f^i(y)|}{K^3} \\
&\leq \frac{LK_2}{K^3} \left( \sum_{i=0}^{n-1} \frac{1}{K^{3i}} \right) |f^{n-1}(x) - f^{n-1}(y)| < \log(2)
\end{aligned}$$

provided  $K$  is sufficiently large.  $\blacksquare$

*Proof of Lemma 3.6.* We first show that  $n(x) > 1$ . Since  $x \in V$ , we have

$$K \geq |f'(x)| = |f''(\gamma_1)| \cdot |x - c|$$

and therefore

$$|f(x) - f(c)| = \frac{1}{2} |f''(\gamma_2)| \cdot |x - c|^2 \leq \frac{|f''(\gamma_2)|}{2|f''(\gamma_1)|^2} K^2 \leq \frac{\|\Phi''\|_{C^0}}{2k_1 L} K^2.$$

We may assume the final quantity is less than  $\frac{1}{4K_2} K^3 L^{-1}$ . If  $n(x) = 2$ , then Sublemma 3.7 implies

$$\frac{1}{4K_2} K^3 L^{-1} < |f^2(x) - f^2(c)| = |(f^2)'(\gamma_3)| \cdot |x - c| \leq 2|(f^2)'(x)| \cdot |x - c|.$$

This inequality coupled with the estimate  $|x - c| \leq \frac{K}{L\|\Phi''\|_{C^0}}$  implies

$$|(f^2)'(x)| > \frac{\|\Phi''\|_{C^0}}{8K_2} K^2.$$

Now assume  $n = n(x) \geq 3$ . Applying Sublemma 3.7 to estimate  $|f^{n-1}(x) - f^{n-1}(c)|$  and  $|f^n(x) - f^n(c)|$ , we have

$$(3.4) \quad \frac{1}{2} |f''(\gamma_2)| \cdot |x - c|^2 \cdot \frac{1}{2} |(f^{n-2})'(f(c))| \leq \frac{1}{4K_2} K^3 L^{-1},$$

$$(3.5) \quad \frac{1}{2} |f''(\gamma_2)| \cdot |x - c|^2 \cdot 2|(f^{n-1})'(f(c))| > \frac{1}{4K_2} K^3 L^{-1}.$$

The recovery estimate follows from the lower bound

$$|(f^n)'(x)| \geq \frac{1}{2} |f''(\gamma_1)| \cdot |x - c| \cdot |(f^{n-1})'(f(c))|.$$

Replacing  $|(f^{n-1})'(f(c))|$  with the lower bound provided by (3.5) and then replacing  $|x - c|^{-1}$  with the lower bound provided by (3.4) yields

$$|(f^n)'(x)| \geq \frac{k_1}{8K_2} K^{\frac{3}{2}n - \frac{3}{2}} \geq \frac{k_1}{8K_2} K^n.$$

$\blacksquare$



We have shown that for  $L$  sufficiently large, if  $f$  satisfies Definition 3.1(B), then  $f \in \mathcal{M}$  with  $V = \{x \in S^1 : |f'(x)| \leq K\}$ . We now find  $a^* \in [0, 2\pi)$  and  $L^* \in [L, L + 2\pi/|\Phi(c^{(2)}) - \Phi(c^{(1)})|)$  such that  $f_{a^*, L^*}$  satisfies Definition 3.1(B) by applying Proposition 3.2. Additional lower bounds on  $K$  will be given as the need arises.

Hypotheses (H1)-(H3) are verified as follows. Let  $z \in S^1$  be such that  $d(z, C) \geq d(y, C)$  for all  $y \in S^1$ . We have  $d(z, C_{\frac{1}{2}\sigma}) \geq \frac{\pi}{2} - \frac{1}{2}\sigma$ . There exists  $\tilde{a} \in [0, 2\pi)$  and  $\tilde{L} \in [L, L + 2\pi/|\Phi(c^{(2)}) - \Phi(c^{(1)})|)$  such that  $\gamma_1^{(1)}(\tilde{a}, \tilde{L}) = \gamma_1^{(2)}(\tilde{a}, \tilde{L}) = z$ . This is so because

$$|\partial_L \gamma_1^{(2)}(a, L) - \partial_L \gamma_1^{(1)}(a, L)| = |\Phi(c^{(2)}) - \Phi(c^{(1)})|.$$

Referring to the setting of Subsections 3.1 and 3.2, we have  $J(a, L) = \Phi(c^{(2)}) - \Phi(c^{(1)})$  and we therefore set  $k_0 = |\Phi(c^{(2)}) - \Phi(c^{(1)})|$ . Let  $\Delta_1$  be a parameter interval in  $a$ -space of length  $\frac{\lambda_M}{k_0^2} K^{-3}$  centered at  $\tilde{a}$  and let  $\Delta_2$  be a parameter interval in  $L$ -space of the same length centered at  $\tilde{L}$ . We assume  $K$  is sufficiently large so that  $\Delta_2 \subset [L, L + 2\pi/|\Phi(c^{(2)}) - \Phi(c^{(1)})|)$ . The image  $\Gamma_1(\Delta_1 \times \Delta_2)$  contains a box such that the length of each of the sides is equal to  $K^{-3}$ . Let  $I_1$  and  $I_2$  be the vertical and horizontal projections of this box onto  $I$ , respectively. Since

$$|\gamma_1^{(i)}(a, L) - z| \leq \max\{1, |\Phi(c^{(1)})|, |\Phi(c^{(2)})|\} \cdot \frac{\lambda_M}{2k_0^2} K^{-3} < \frac{\pi}{2} - \frac{1}{2}\sigma$$

for  $i \in \{1, 2\}$  and for all  $(a, L) \in \Delta_1 \times \Delta_2$  provided  $K$  is sufficiently large, we have  $I_1 \subset S^1 \setminus C_{\frac{1}{2}\sigma}$  and  $I_2 \subset S^1 \setminus C_{\frac{1}{2}\sigma}$ .

By construction, (H1) is satisfied with  $n_0 = 1$  and the intervals  $I_1$  and  $I_2$  satisfy (H3). Setting  $\varepsilon_1 = 1$ , (H2) is satisfied because  $|f'_{a, L}| \geq K^3$  on  $S^1 \setminus C_{\frac{1}{2}\sigma}$  for all  $(a, L) \in \Delta_1 \times \Delta_2$ . If  $K$  is large enough so that (3.1) holds, then the application of Proposition 3.2 with  $\delta_1 = \frac{1}{2}\sigma$  produces a Misiurewicz pair  $(a^*, L^*) \in \Delta_1 \times \Delta_2$ . ■

#### 4. THEORY OF RANK ONE ATTRACTORS

Let  $D$  denote the closed unit disk in  $\mathbb{R}^{n-1}$  and let  $M = S^1 \times D$ . We consider a family of maps  $T_{a, b} : M \rightarrow M$ , where  $\mathbf{a} = (a_1, \dots, a_k) \in \Omega$  is a vector of parameters and  $b \in B_0$  is a scalar parameter. Here  $\Omega = \Omega_1 \times \dots \times \Omega_k \subset \mathbb{R}^k$  is a product of intervals and  $B_0 \subset \mathbb{R} \setminus \{0\}$  is a subset of  $\mathbb{R}$  with an accumulation point at 0. Points in  $M$  are denoted by  $(x, y)$  with  $x \in S^1$  and  $y \in D$ . Rank one theory postulates the following.

##### (G1) Regularity conditions.

- (a) For each  $b \in B_0$ , the function  $(x, y, \mathbf{a}) \mapsto T_{a, b}(x, y)$  is  $C^3$ .
- (b) Each map  $T_{a, b}$  is an embedding of  $M$  into itself.
- (c) There exists  $K_D > 0$  independent of  $\mathbf{a}$  and  $b$  such that for all  $\mathbf{a} \in \Omega$ ,  $b \in B_0$ , and  $z, z' \in M$ , we have

$$\frac{|\det DT_{a, b}(z)|}{|\det DT_{a, b}(z')|} \leq K_D.$$

- (G2) **Existence of a singular limit.** For  $\mathbf{a} \in \Omega$ , there exists a map  $T_{a, 0} : M \rightarrow S^1 \times \{0\}$  such that the following holds. We select a special index  $j \in \{1, \dots, k\}$ . For every fixed set  $\{a_i \in \Omega_i : i \neq j\}$ , the maps  $(x, y, a_j) \mapsto T_{a, b}(x, y)$  converge in the  $C^3$  topology to  $(x, y, a_j) \mapsto T_{a, 0}(x, y)$ . Identifying  $S^1 \times \{0\}$  with  $S^1$ , we refer to  $T_{a, 0}$  and the restriction  $f_a : S^1 \rightarrow S^1$  defined by  $f_a(x) = T_{a, 0}(x, 0)$  as the *singular limit* of  $T_{a, b}$ .

**(G3) Existence of a sufficiently expanding map within the singular limit.**

There exists  $\mathbf{a}^* = (a_1^*, \dots, a_k^*) \in \Omega$  such that  $f_{\mathbf{a}^*} \in \mathcal{M}$ .

**(G4) Parameter transversality.** Let  $C_{\mathbf{a}^*}$  denote the critical set of  $f_{\mathbf{a}^*}$ . Define  $\tilde{\mathbf{a}}_j = (a_1^*, \dots, a_{j-1}^*, a_j, a_{j+1}^*, \dots, a_k^*)$ . We say that the family  $\{f_{\mathbf{a}}\}$  satisfies the **parameter transversality** condition with respect to parameter  $a_j$  if the following holds. For each  $x \in C_{\mathbf{a}^*}$ , let  $p = f(x)$  and let  $x(\tilde{\mathbf{a}}_j)$  and  $p(\tilde{\mathbf{a}}_j)$  denote the continuations of  $x$  and  $p$ , respectively, as the parameter  $a_j$  varies around  $a_j^*$ . The point  $p(\tilde{\mathbf{a}}_j)$  is the unique point such that  $p(\tilde{\mathbf{a}}_j)$  and  $p$  have identical itineraries under  $f_{\tilde{\mathbf{a}}_j}$  and  $f_{\mathbf{a}^*}$ , respectively. We have

$$\left. \frac{d}{da_j} f_{\tilde{\mathbf{a}}_j}(x(\tilde{\mathbf{a}}_j)) \right|_{a_j=a_j^*} \neq \left. \frac{d}{da_j} p(\tilde{\mathbf{a}}_j) \right|_{a_j=a_j^*}.$$

**(G5) Nondegeneracy at ‘turns’.** For each  $x \in C_{\mathbf{a}^*}$ , there exists  $1 \leq \ell \leq n-1$  such that

$$\frac{\partial}{\partial y_\ell} T_{\mathbf{a}^*, 0}(x, 0) \neq 0.$$

**(G6) Conditions for mixing.**

- (a) We have  $e^{\frac{1}{3}\lambda_0} > 2$ , where  $\lambda_0$  is defined within Definition 3.1.
- (b) Let  $J_1, \dots, J_r$  be the intervals of monotonicity of  $f_{\mathbf{a}^*}$ . Let  $Q = (q_{ij})$  be the matrix defined by

$$q_{ij} = \begin{cases} 1, & \text{if } f_{\mathbf{a}^*}(J_i) \supset J_j, \\ 0, & \text{otherwise.} \end{cases}$$

There exists  $N > 0$  such that  $Q^N > 0$ .

The following lemma often facilitates the verification of **(G4)**.

**Lemma 4.1** ([5, 4]). *Let  $f = f_{\mathbf{a}^*}$ . Suppose that for all  $x \in C_{\mathbf{a}^*}$ , we have*

$$\sum_{k=0}^{\infty} \frac{1}{|(f^k)'(f(x))|} < \infty.$$

*Then for each  $x \in C_{\mathbf{a}^*}$ ,*

$$(4.1) \quad \sum_{k=0}^{\infty} \frac{[(\partial_{a_j} f_{\tilde{\mathbf{a}}_j})(f^k(x))]_{a_j=a_j^*}}{(f^k)'(f(x))} = \left[ \frac{d}{da_j} f_{\tilde{\mathbf{a}}_j}(x(\tilde{\mathbf{a}}_j)) - \frac{d}{da_j} p(\tilde{\mathbf{a}}_j) \right]_{a_j=a_j^*}.$$

Rank one theory states that given a family  $\{T_{\mathbf{a},b}\}$  satisfying **(G1)**-**(G5)**, a measure-theoretically significant subset of this family consists of maps admitting attractors with strong chaotic and stochastic properties. We formulate the precise results and we then describe the properties that the attractors possess.

**Theorem 4.2** ([8, 7]). *Suppose the family  $\{T_{\mathbf{a},b}\}$  satisfies **(G1)**-**(G3)** and **(G5)**. For all  $1 \leq j \leq k$  such that the parameter  $a_j$  satisfies **(G4)** and for all sufficiently small  $b \in B_0$ , there exists a subset  $A_j \subset \Omega_j$  of positive Lebesgue measure such that for  $a_j \in A_j$ ,  $T_{\tilde{\mathbf{a}}_j,b}$  admits a strange attractor  $\Lambda$  with properties **(SA1)**, **(SA2)**, and **(SA3)**.*

**Theorem 4.3** ([8, 9, 7]). *In the sense of Theorem 4.2,*

$$(G1)-(G6) \implies (SA1)-(SA4).$$

- (SA1) **Positive Lyapunov exponent.** Let  $U$  denote the basin of attraction of the attractor  $\Lambda$ . For almost every  $(x, y) \in U$  with respect to Lebesgue measure, the orbit of  $(x, y)$  has a positive Lyapunov exponent. That is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|DT^n(x, y)\| > 0.$$

- (SA2) **Existence of SRB measures and basin property.**

- (a) The map  $T$  admits at least one and at most finitely many ergodic SRB measures all of which have no zero Lyapunov exponents. Let  $\nu_1, \dots, \nu_r$  denote these measures.
- (b) For Lebesgue-a.e.  $(x, y) \in U$ , there exists  $j(x) \in \{1, \dots, r\}$  such that for every continuous function  $\varphi : U \rightarrow \mathbb{R}$ ,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i(x, y)) \rightarrow \int \varphi d\nu_{j(x)}.$$

- (SA3) **Statistical properties of dynamical observations.**

- (a) For every ergodic SRB measure  $\nu$  and every Hölder continuous function  $\varphi : \Lambda \rightarrow \mathbb{R}$ , the sequence  $\{\varphi \circ T^i : i \in \mathbb{Z}^+\}$  obeys a central limit theorem. That is, if  $\int \varphi d\nu = 0$ , then the sequence

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^i$$

converges in distribution to the normal distribution. The variance of the limiting normal distribution is strictly positive unless  $\varphi \circ T = \psi \circ T - \psi$  for some  $\psi$ .

- (b) Suppose that for some  $N \geq 1$ ,  $T^N$  has an SRB measure  $\nu$  that is mixing. Then given a Hölder exponent  $\eta$ , there exists  $\tau = \tau(\eta) < 1$  such that for all Hölder  $\varphi, \psi : \Lambda \rightarrow \mathbb{R}$  with Hölder exponent  $\eta$ , there exists  $L = L(\varphi, \psi)$  such that for all  $n \in \mathbb{N}$ ,

$$\left| \int (\varphi \circ T^{nN}) \psi d\nu - \int \varphi d\nu \int \psi d\nu \right| \leq L(\varphi, \psi) \tau^n.$$

- (SA4) **Uniqueness of SRB measures and ergodic properties.**

- (a) The map  $T$  admits a unique (and therefore ergodic) SRB measure  $\nu$ , and
- (b) the dynamical system  $(T, \nu)$  is mixing, or, equivalently, isomorphic to a Bernoulli shift.

## 5. PROOF OF THEOREM 2.1

**5.1. Degenerate Hopf bifurcation: the reduced equations.** We study the two-dimensional system

$$(5.1) \quad \begin{cases} \dot{r} = -\mu r \\ \dot{\theta} = \omega + \gamma_\mu \mu + \beta_\mu r^2 \end{cases}$$

System (5.1) is obtained from (2.1) by setting  $g_\mu = h_\mu = 0$ . Let  $F_t$  denote the flow of (5.1). For  $\mu$  sufficiently large,  $F_{\tau(\mu)} \circ \kappa$  maps  $\mathcal{A}$  into  $\mathcal{A}$ . The study of this annulus map is the central goal of this subsection.

We introduce a new coordinate system in order to standardize the position and size of  $\mathcal{A}$ . Let  $r = \mu^{\rho_1} z$ . Written in terms of  $z$  and  $\theta$ , system (5.1) becomes

$$(5.2) \quad \begin{cases} \dot{z} = -\mu z \\ \dot{\theta} = \omega + \gamma_\mu \mu + \mu^{2\rho_1} \beta_\mu z^2 \end{cases}$$

Let  $G_t$  denote the flow associated with (5.2). The kick map  $\kappa$  is now given in rectangular coordinates by

$$\kappa \begin{pmatrix} z \cos(\theta) \\ z \sin(\theta) \end{pmatrix} = \begin{pmatrix} z \cos(\theta) \\ z \sin(\theta) + L\mu^{\rho_2 - \rho_1} \end{pmatrix}$$

We have  $\mathcal{A} = \{(z, \theta) : K_4^{-1} \leq z \leq K_4\}$ . The relaxation time  $\tau(\mu)$  is given by

$$(5.3) \quad \tilde{z} e^{-\mu\tau(\mu)} = 1,$$

where  $\tilde{z} = L\mu^{\rho_2 - \rho_1} - K_4$ . Let  $\Psi_\mu = G_{\tau(\mu)} \circ \kappa$ . For  $\mu$  sufficiently large,  $\Psi_\mu$  maps  $\mathcal{A}$  into  $\mathcal{A}$ . We now derive  $\Psi_\mu : \mathcal{A} \rightarrow \mathcal{A}$  explicitly.

Let  $(z_0, \theta_0) \in \mathcal{A}$ . Writing  $\kappa(z_0, \theta_0) = (z_1, \theta_1)$ , we have

$$(5.4) \quad \begin{aligned} z_1^2 &= z_0^2 + 2L\mu^{\rho_2 - \rho_1} z_0 \sin(\theta_0) + L^2 \mu^{2(\rho_2 - \rho_1)}, \\ \theta_1 &= \frac{\pi}{2} - \tan^{-1} \left( \frac{z_0 \cos(\theta_0)}{z_0 \sin(\theta_0) + L\mu^{\rho_2 - \rho_1}} \right). \end{aligned}$$

Integrating (5.2) and writing  $G_t \circ \kappa = (z(t), \theta(t))$ , we have

$$\begin{aligned} z(t) &= z_1 e^{-\mu t}, \\ \theta(t) &= \theta_1 + t(\omega + \gamma_\mu \mu) + \frac{\beta_\mu}{2} \mu^{2\rho_1 - 1} z_1^2 (1 - e^{-2\mu t}). \end{aligned}$$

Evaluating  $\theta(\tau(\mu))$  using (5.3), we have

$$(5.5) \quad \theta(\tau(\mu)) = \theta_1 + (\omega + \gamma_\mu \mu)\tau(\mu) + \frac{\beta_\mu}{2} \mu^{2\rho_1 - 1} \left( z_1^2 - \frac{z_1^2}{\tilde{z}^2} \right).$$

Replacing the first occurrence of  $z_1^2$  in (5.5) with the right side of (5.4), we obtain

$$\theta(\tau(\mu)) = \theta_1 + \xi(\mu) + \frac{\beta_\mu}{2} \left( \mu^{2\rho_1 - 1} z_0^2 + 2Lz_0 \sin(\theta_0) \mu^{\rho_1 + \rho_2 - 1} - \mu^{2\rho_1 - 1} \frac{z_1^2}{\tilde{z}^2} \right),$$

where

$$\xi(\mu) = (\omega + \gamma_\mu \mu)\tau(\mu) + \frac{\beta_\mu}{2} L^2 \mu^{2\rho_2 - 1}.$$

The second component of  $\Psi_\mu$  is given by

$$z(\tau(\mu)) = \frac{z_1}{\tilde{z}}.$$

We wish to show that the family  $\{\Psi_\mu\}$  converges to a singular limit as  $\mu \rightarrow 0$ . This cannot be accomplished directly because  $\xi(\mu)$  diverges as  $\mu \rightarrow 0$ , preventing the convergence of  $\theta(\tau(\mu))$ . We overcome this difficulty by taking advantage of the fact that  $\theta(t)$  is computed modulo  $2\pi$ . Assume that  $\omega > 0$ . For  $\mu$  sufficiently small,  $\xi(\mu)$  is monotone. In addition,  $\xi(\mu) \rightarrow \infty$  as  $\mu \rightarrow 0$ . Let  $(\mu_n)$  be a sequence such that  $\mu_n \rightarrow 0$  monotonically,  $\xi$  is monotone on  $(0, \mu_1]$ , and  $\xi(\mu_n) \in 2\pi\mathbb{Z}$  for all  $n \in \mathbb{N}$ . We introduce the parameter  $a \in [0, 2\pi)$  and write  $\Psi_\mu$  in terms of  $a$ . For  $n \in \mathbb{N}$  and  $a \in [0, 2\pi)$ , let  $\mu(a, n) = \xi^{-1}(\xi(\mu_n) + a)$ . When referring to  $\mu(a, n)$ , we will

henceforth suppress the dependence on  $n$  and simply write  $\mu(a)$ . The problematic term  $\xi(\mu)$  becomes  $\xi(\mu(a)) = a$ . Writing  $\Psi_{\mu(a)} = T_{a,L,\mu_n}$ , we have

$$\begin{aligned} T_{a,L,\mu_n}^{(1)}(z_0, \theta_0) &= \frac{z_1}{\tilde{z}}, \\ T_{a,L,\mu_n}^{(2)}(z_0, \theta_0) &= \theta_1 + a + \frac{\beta_{\mu(a)}}{2} \left( \mu(a)^{2\rho_1-1} z_0^2 + \right. \\ &\quad \left. 2L\mu(a)^{\rho_1+\rho_2-1} z_0 \sin(\theta_0) - \mu(a)^{2\rho_1-1} \frac{z_1^2}{\tilde{z}^2} \right), \end{aligned}$$

where  $T^{(1)}$  and  $T^{(2)}$  are the components of  $T$ . Let  $\rho_1$  and  $\rho_2$  satisfy  $\frac{1}{2} < \rho_1 < 1$  and  $\rho_1 + \rho_2 = 1$ . Then as  $n \rightarrow \infty$ ,  $T_{a,L,\mu_n}$  converges in the  $C^0$  topology to the map  $T_{a,L,0}$  defined by

$$\begin{aligned} T_{a,L,0}^{(1)} &= 1 \\ T_{a,L,0}^{(2)} &= \frac{\pi}{2} + \beta_0 L z_0 \sin(\theta_0) + a. \end{aligned}$$

The following lemma asserts that the convergence is strong enough for the application of rank one theory.

**Lemma 5.1.** *Fix  $L > 0$ . The maps  $(z_0, \theta_0, a) \mapsto T_{a,L,\mu_n}(z_0, \theta_0)$  converge in the  $C^3$  topology to the map  $(z_0, \theta_0, a) \mapsto T_{a,L,0}(z_0, \theta_0)$  as  $n \rightarrow \infty$  on the domain  $\mathcal{A} \times [0, 2\pi)$ .*

*Proof of Lemma 5.1.* Holding  $a$  fixed, the derivatives of  $\theta_1$ ,  $\frac{z_1}{\tilde{z}}$ , and  $\frac{z_1^2}{\tilde{z}^2}$  of orders 1, 2, and 3 with respect to  $z_0$  and  $\theta_0$  are  $\mathcal{O}(\mu^{\rho_1-\rho_2})$ . When differentiating with respect to  $a$ , use the fact that for  $i = 1, 2, 3$ ,

$$\partial_a^{(i)} \mu(a) = \mathcal{O} \left( \frac{\mu_n^{i+1}}{(\log(\mu_n^{-1}))^i} \right).$$

■

We finish this subsection with a distortion estimate.

**Lemma 5.2** (Distortion estimate). *Let  $0 < L_2 < L_3$ . There exists  $K_D > 0$  such that for all  $n \in \mathbb{N}$ ,  $a \in [0, 2\pi)$ ,  $L \in [L_2, L_3]$ , and  $(z_0, \theta_0), (z'_0, \theta'_0) \in \mathcal{A}$ , we have*

$$\frac{|\det DT_{a,L,\mu_n}(z_0, \theta_0)|}{|\det DT_{a,L,\mu_n}(z'_0, \theta'_0)|} \leq K_D.$$

*Proof of Lemma 5.2.* Recall that  $T_{a,L,\mu_n} = G_{\tau(\mu(a))} \circ \kappa$ . We bound the distortion by analyzing  $G$  and  $\kappa$  independently. Let  $(z_0, \theta_0), (z'_0, \theta'_0) \in \mathcal{A}$ . Writing  $\mu = \mu(a)$ ,  $\det D\kappa(z_0, \theta_0)$  is given by

$$\frac{z_0^3 + \mu^{\rho_2-\rho_1} \cdot L z_0(1+z_0) \sin(\theta_0) + \mu^{2(\rho_2-\rho_1)} \cdot L^2(1+(z_0-1)\cos^2(\theta_0))}{z_1((z_0 \sin(\theta_0) + L\mu^{\rho_2-\rho_1})^2 + z_0^2 \cos^2(\theta_0))}.$$

This explicit formula implies the estimate

$$\frac{\det D\kappa(z_0, \theta_0)}{\det D\kappa(z'_0, \theta'_0)} = \mathcal{O}(1).$$

Now set  $G = G_{\tau(\mu(a))}$ . For any point in  $\kappa(\mathcal{A})$ , the determinant of the derivative of  $G$  is precisely  $\tilde{z}^{-1}$ . Therefore,

$$\frac{\det DG(z_1, \theta_1)}{\det DG(z'_1, \theta'_1)} = 1.$$

■

**5.2. Inclusion of the higher-order terms in the normal form.** We show that the inclusion of the higher-order terms in the differential equations defining the flow does not affect the form of the singular limit derived in Subsection 5.1. Set  $r = \mu^{\rho_1} \hat{z}$  and  $\theta = \hat{\theta}$ . Written in terms of  $\hat{z}$  and  $\hat{\theta}$ , the normal form (2.1) becomes

$$(5.6) \quad \begin{cases} \dot{\hat{z}} = -\mu \hat{z} + \mu^{4\rho_1} \hat{z}^5 g_\mu(\mu^{\rho_1} \hat{z}, \hat{\theta}) \\ \dot{\hat{\theta}} = \omega + \gamma_\mu \mu + \beta_\mu \mu^{2\rho_1} \hat{z}^2 + \mu^{4\rho_1} \hat{z}^4 h_\mu(\mu^{\rho_1} \hat{z}, \hat{\theta}) \end{cases}$$

Let  $\widehat{G}_t$  denote the flow generated by (5.6). We define the family  $\{\widehat{T}\}$  on  $\mathcal{A}$  by first applying the kick map  $\kappa$  and then allowing the  $\widehat{G}_t$ -flow to return  $\kappa(\mathcal{A})$  to  $\mathcal{A}$ . Set  $\widehat{T}_{a,L,\mu_n} = \widehat{G}_{\tau(\mu(a))} \circ \kappa$ .

**Lemma 5.3.** *Fix  $L > 0$ . If  $\rho_2 > \frac{1}{3}$ , then the maps  $(z_0, \theta_0, a) \mapsto \widehat{T}_{a,L,\mu_n}(z_0, \theta_0)$  converge in the  $C^3$  topology to the map  $(z_0, \theta_0, a) \mapsto T_{a,L,0}(z_0, \theta_0)$  as  $n \rightarrow \infty$  on the domain  $\mathcal{A} \times [0, 2\pi)$ .*

*Proof of Lemma 5.3.* Computing the first component of  $\widehat{G}_t \circ \kappa$ , we have

$$\begin{aligned} \hat{z}(t) &= z_1 e^{-\mu t} \left( 1 + \mu^{4\rho_1} \int_0^t z_1^{-1} e^{\mu s} \hat{z}(s)^5 g_\mu(\mu^{\rho_1} \hat{z}(s), \hat{\theta}(s)) ds \right) \\ &= z(t) + \zeta(t), \end{aligned}$$

where the perturbative term  $\zeta(t)$  is defined by

$$(5.7) \quad \zeta(t) = \mu^{4\rho_1} e^{-\mu t} \int_0^t e^{\mu s} \hat{z}(s)^5 g_\mu(\mu^{\rho_1} \hat{z}(s), \hat{\theta}(s)) ds.$$

Computing the second component of  $\widehat{G}_t \circ \kappa$ , we have  $\hat{\theta}(t) = \theta(t) + \tilde{\theta}(t)$ , where

$$\begin{aligned} (5.8) \quad \tilde{\theta}(t) &= \beta_\mu \mu^{2\rho_1} \int_0^t \left[ 2\mu^{4\rho_1} z_1 e^{-2\mu v} \int_0^v e^{\mu s} \hat{z}(s)^5 g_\mu(\mu^{\rho_1} \hat{z}(s), \hat{\theta}(s)) ds \right. \\ &\quad \left. + \mu^{8\rho_1} e^{-2\mu v} \left( \int_0^v e^{\mu s} \hat{z}(s)^5 g_\mu(\mu^{\rho_1} \hat{z}(s), \hat{\theta}(s)) ds \right)^2 \right] dv \\ &\quad + \mu^{4\rho_1} \int_0^t \hat{z}(s)^4 h_\mu(\mu^{\rho_1} \hat{z}(s), \hat{\theta}(s)) ds. \end{aligned}$$

In order to establish  $C^0$  convergence, it suffices to show that the perturbative terms  $\zeta(\tau(\mu(a)))$  and  $\tilde{\theta}(\tau(\mu(a)))$  converge to 0 in the  $C^0$  topology as  $n \rightarrow \infty$ . Estimating the integrals in (5.7) and (5.8), we obtain

$$\begin{aligned} \zeta(\tau(\mu)) &= \mathcal{O}(\mu^{5\rho_2 - \rho_1 - 1} \log(\mu^{-1})), \\ \tilde{\theta}(\tau(\mu)) &= \mathcal{O}(\mu^{6\rho_2 - 2} (\log(\mu^{-1}))^2) + \mathcal{O}(\mu^{10\rho_2 - 3} (\log(\mu^{-1}))^3). \end{aligned}$$

Since  $\rho_2 \in (\frac{1}{3}, \frac{1}{2})$  and  $\rho_1 \in (\frac{1}{2}, \frac{2}{3})$ , we have

$$\|\zeta(\tau(\mu))\|_{C^0} \rightarrow 0 \text{ and } \|\tilde{\theta}(\tau(\mu))\|_{C^0} \rightarrow 0$$

as  $\mu \rightarrow 0$ .

We complete the proof of Lemma 5.3 by showing that

$$\|D^i \widehat{G}_{\tau(\mu(a))} \circ \kappa - D^i G_{\tau(\mu(a))} \circ \kappa\|_{C^0} \rightarrow 0$$

for  $1 \leq i \leq 3$ . In light of Lemma 5.1, this establishes the asserted  $C^3$  convergence. Since  $\|D^i \kappa\|_{C^0}$  is bounded for  $1 \leq i \leq 3$ , it is sufficient to show that  $\|D^i \widehat{G}_{\tau(\mu(a))} - D^i G_{\tau(\mu(a))}\|_{C^0} \rightarrow 0$  for  $1 \leq i \leq 3$ . We use the following elementary Gronwall-type lemma.

**Lemma 5.4** ([10]). *Let  $\Lambda \subset \mathbb{R}^N$  be a convex open domain. Let  $W$  and  $\widehat{W}$  be  $C^1$  vector fields on  $\Lambda$ . Suppose that for  $t \in [0, t_0]$ ,  $\hat{\varphi}$  and  $\varphi$  solve the equations*

$$\frac{d\hat{\varphi}}{dt} = \widehat{W}(\hat{\varphi}) \text{ and } \frac{d\varphi}{dt} = W(\varphi)$$

with  $\hat{\varphi}(0) = \varphi(0)$ . Then for all  $t \in [0, t_0]$ , we have

$$\|\hat{\varphi}(t) - \varphi(t)\| \leq \frac{A_1}{A_2}(e^{A_2 t} - 1),$$

where

$$A_1 = \sup_{x \in \Lambda} \|\widehat{W}(x) - W(x)\| \text{ and } A_2 = \sum_{j=1}^N \sup_{x \in \Lambda} \|DW^{(j)}(x)\|.$$

We rescale time in (5.2) and (5.6) by setting  $t = t' \tau(\mu(a))$ . Let  $\eta$  and  $\hat{\eta}$  denote the rescaled vector fields. We have

$$\begin{cases} \eta^{(1)} = \tau(\mu(a))(-\mu z) \\ \eta^{(2)} = \tau(\mu(a))(\omega + \gamma_\mu \mu + \mu^{2\rho_1} \beta_\mu z^2) \end{cases}$$

$$\begin{cases} \hat{\eta}^{(1)} = \tau(\mu(a))(-\mu \hat{z} + \mu^{4\rho_1} \hat{z}^5 g_\mu(\mu^{\rho_1} \hat{z}, \hat{\theta})) \\ \hat{\eta}^{(2)} = \tau(\mu(a))(\omega + \gamma_\mu \mu + \beta_\mu \mu^{2\rho_1} \hat{z}^2 + \mu^{4\rho_1} \hat{z}^4 h_\mu(\mu^{\rho_1} \hat{z}, \hat{\theta})) \end{cases}$$

We explicitly treat the case  $i = 1$ . The cases  $i = 2$  and  $i = 3$  are handled using the same technique. Apply Lemma 5.4 with  $\hat{\varphi} = D\widehat{G}$ ,  $\varphi = DG$ ,  $\widehat{W} = D\hat{\eta}$ ,  $W = D\eta$ , and  $t = 1$ . The quantity  $A_2$  is bounded. Therefore, the estimate  $A_1 = \mathcal{O}(\mu^{5\rho_2 - \rho_1 - 1} \log(\mu^{-1}))$  implies that

$$(5.9) \quad \|D\widehat{G}_1 - DG_1\|_{C^0} = \mathcal{O}(\mu^{5\rho_2 - \rho_1 - 1} \log(\mu^{-1})).$$

■

**5.3. Verification of (G1)-(G6).** Theorem 2.1 follows from an application of Theorems 4.2 and 4.3. Statements (1) and (2) of Theorem 2.1 require the verification of (G1)-(G5) for the family  $\{\widehat{T}_{a,L,\mu_n}\}$ . Statement (3) of Theorem 2.1 requires the additional verification of (G6).

We proceed with the verification of statement (1) of Theorem 2.1. Properties (G1)(a) and (G1)(b) follow from the general theory of ordinary differential equations. For (G1)(c), it suffices to show that the distortion of  $\widehat{G}_{\tau(\mu(a))}$  is bounded because the distortion of  $\kappa$  is bounded. Using (5.9), we have

$$D\widehat{G}_{\tau(\mu(a))}(z_1, \theta_1) = \begin{pmatrix} \tilde{z}^{-1} + \varepsilon_1 & \varepsilon_2 \\ \frac{\beta_\mu}{2} \mu^{2\rho_1 - 1} (2z_1 - \frac{2z_1}{\tilde{z}^2}) + \varepsilon_3 & 1 + \varepsilon_4 \end{pmatrix}$$

where  $\varepsilon_j = \mathcal{O}(\mu^{5\rho_2 - \rho_1 - 1} \log(\mu^{-1}))$  for  $1 \leq j \leq 4$ . Since  $\rho_2 > \frac{3}{8}$ ,  $\frac{\varepsilon_j}{\tilde{z}^{j-1}} \rightarrow 0$  as  $\mu \rightarrow 0$  for  $1 \leq j \leq 4$ . Therefore, we have

$$\det(D\widehat{G}_{\tau(\mu(a))}(z_1, \theta_1)) = \tilde{z}^{-1} + \mathcal{O}(\mu^{5\rho_2 - \rho_1 - 1} \log(\mu^{-1})).$$

This estimate implies that the distortion of  $\widehat{G}_{\tau(\mu(a))}$  is bounded.

Lemma 5.3 establishes **(G2)**. Let  $f_{a,L}$  denote the restriction of  $T_{a,L,0}^{(2)}$  to the circle  $S^1 = \{(z_0, \theta_0) : z_0 = 1\}$ . We have

$$f_{a,L}(\theta) = \frac{\pi}{2} + \beta_0 L \sin(\theta) + a.$$

Applying Theorem 3.4 with  $\Phi(\theta) = \sin(\theta)$ ,  $c^{(1)} = \frac{\pi}{2}$ , and  $c^{(2)} = \frac{3\pi}{2}$ , if  $L$  is sufficiently large then there exist  $L^* \in [L, L + \frac{\pi}{|\beta_0|}]$  and  $a^* \in [0, 2\pi)$  such that  $f_{a^*,L^*} \in \mathcal{M}$ . This is **(G3)**. We establish parameter transversality **(G4)** by applying Lemma 4.1. Write  $f = f_{a^*,L^*}$  and  $f_a = f_{a,L^*}$ . We have  $\partial_a f_a(\cdot) = 1$  and  $|(f^k)'(f(x))| \geq K^k$ . Therefore, the absolute value of the left side of (4.1) is bounded below by  $1 - \sum_{k=1}^{\infty} K^{-k}$ . This quantity is positive if  $K > 2$ . For **(G5)**, observe that

$$\partial_{z_0} T_{a,L,0}^{(2)}(1, c^{(1)}) = \beta_0 L \neq 0 \text{ and } \partial_{z_0} T_{a,L,0}^{(2)}(1, c^{(2)}) = -\beta_0 L \neq 0.$$

This completes the verification of statement (1) of Theorem 2.1.

Statement (2) of Theorem 2.1 follows from the fact that for all  $L$  sufficiently large,  $f_{a,L} \in \mathcal{M}$  for a  $\mathcal{O}(L^{-1})$ -dense set of values of  $a$ . Wang and Young [9] prove this result in a slightly different context. The proof for the family  $\{f_{a,L}\}$  is essentially the same.

Statement (3) of Theorem 2.1 requires the verification of the conditions for mixing **(G6)**. Property **(G6)(a)** holds provided  $e^{\lambda_0} = K > 8$ . Property **(G6)(b)** is satisfied with  $N = 1$  provided  $L$  is sufficiently large.

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